CALCULUS IN MOTION™
presents
ALGEBRA IN MOTION™

DYNAMIC ANIMATIONS OF ALGEBRA
Classroom Ready / No GSP Experience Necessary
Windows / Macintosh (Requires Geometer’s Sketchpad v4)

COORDINATE PLANE
ELEMENTARY GRAPHING
MONOMIAL AND BINOMIAL MULTIPLICATION
PYTHAGOREAN THEOREM
INTRIGUING REAL-WORLD SITUATIONS

ABSOLUTE VALUE SENTENCES
ANIMATION OF SLOPE’S RISE/RUN RATIO
LINEAR EQUATIONS, INEQUALITIES, SYSTEMS
QUADRATIC FUNCTIONS
DISTANCE AND MIDPOINT FORMULAS
EXPANSION AND FACTORING OF POLYNOMIALS

CONICS – DEFINITIONS, GRAPHS, APPLICATIONS
GRAPH A POINT IN 3D-SPACE
GRAPH POINTS ON THE COMPLEX PLANE
GREATEST INTEGER FUNCTIONS
POLYNOMIAL “EVOLUTION”
LOGARITHMIC AND EXPONENTIAL FUNCTIONS
INVERSES

“MORPH” PERIOD AND AMPLITUDE
UNIT CIRCLE “UNWRAPPED”
UNIT CIRCLE’S ANGLES, COORDINATES, AND RATIOS
PYTHAGOREAN IDENTITIES
LAW OF SINES AND LAW OF COSINES

ALL THE CLASSIC FUNCTIONS
PARAMETRIC AND POLAR GRAPHING
ASTONISHING PROPERTIES OF POLYNOMIALS
CLASSIC “OPEN BOX” PROBLEM

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www.calculusinmotion.com ~ Audrey Weeks ~ amweeks@aol.com ~Tel/Fax: 818 845-6332
## CONNECTING THE ANIMATIONS TO MATH COURSES

Many files contain several separate animations accessed by tabs at the bottom of the GSP screen. Always maximize your GSP screen.

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GENERAL DIRECTIONS FOR ALL FILES

1. THESE FILES MUST BE ACCESSED FROM GEOMETER’S SKETCHPAD (GSP) V4 (NO PRIOR VERSION).
   Launch GSP4 first and access these files by pulling down the “File” -> “Open” menu from within GSP4.

2. MAC USERS: You may see the following message when opening files. Ignore it.
   This document contains some supplemental information that cannot be read by this version of Sketchpad. It will open as a copy.

3. SCREEN SIZE: It is best if your monitor screen size is set to 1024x768. Always maximize the GSP screen and each GSP file to see other links (tabs) at the bottom of most animations. Also, be sure your text palette does not appear. It is an optional toolbar at the bottom of your screen that will obscure some items. To hide the text palette, access the drop-down menu called “Display” and release on “Hide Text Palette”.

4. UNDO: To undo an unintentional action, access the drop-down menu “Edit” and release on “Undo”.

5. TO DE-SELECT HIGHLIGHTED ITEMS: Click anywhere on the clear background. (If too many items are selected at once, actions may not occur as desired.)

6. DRAGGING AN OBJECT: (two options)
   - Click-hold-drag-release
   - Highlight it (by clicking once on it) then press the left-right-up-down arrows on your keyboard. This is great for moving one pixel at a time.
   When dragging items, keep in mind that due to the discrete nature of a “pixelized” plane, you may not be able to land exactly on the coordinates you desire. Please know that every attempt has been made to allow entry of specific values and/or integer access where that was found to be desirable.

7. PAGE TABS: Most of these files contain multiple animations. Access each animation’s page from the small labeled tabs at the bottom left of the GSP screen.

8. SLIDERS: Many of these animations contain sliders to “morph” the drawn curve (they control the values of the coefficients of the function). When present, they generally appear at the lower left of the screen on truncated number lines. They are extremely powerful visual tools. Simply drag the slider points to see what happens. To rescale the sliders, drag the red tick mark at 1.

9. MOTION SPEED: If the motion is too fast or too slow, go to “Display” -> “Show Motion Controller”. These controls allow the motion to change speed, reverse, pause, and stop.

10. BUTTONS: In all files, to explore the animations, simply press (click) each of pre-made buttons one at a time from top to bottom (and left to right). You need not stop an animation before continuing to press other buttons or drag the sliders. To stop a button’s action, re-press that button (it will appear darkened while it is running), or press ESC on your keyboard. Mix and match any combination of button presses as desired.

11. DOMAINS: They appear as bold segments on the x-axis. Adjust them by dragging their endpoints.

12. ARROWS ON CURVES: If they are visible, they may be dragged to reveal more or less of the plotted curve.

13. ESC: Pressing ESC on your keyboard (repeatedly) will stop movement, then deselect objects one at a time, then erase traces.

Rounded decimals can be misleading: When measuring, GSP uses a large number of digits but rounds off the displayed values. This may at times cause a calculation to appear to be off by one unit in the last digit. For example, if 2.3 + 1.4 = 3.6 appears, the numbers may have been 2.26….. + 1.37….. which is 3.63…..

*****DIRECTIONS FOR EACH ANIMATION FOLLOW *****
“Abs Val Inequalities”

SOLVING INEQUALITIES FOR \( x \) CONTAINING \(|ax+b|\)

This animation empowers students to use the meaning of absolute value (a distance from zero on a number line) to create the corresponding compound sentence that makes sense. For instance, when solving \(|2x + 1| > 5\), the sentence itself tells us that the expression \(2x+1\) must be positioned more than 5 units from zero. The animation demonstrates this visual connection and solves the problem.

How to run this animation:

1. The opening values are \(a=2, b=1, c=5\). Initially, you may want to keep these values. All values will be rounded to 3 decimal places.
2. Choose to solve any of the following inequalities by pressing the corresponding button at the left side of the screen:

\[
|ax + b| = c \quad |ax + b| > c \quad |ax + b| < c \quad |ax + b| \geq c \quad |ax + b| \leq c
\]

This is what happens:

a) Your chosen equation or inequality appears at the top of the screen with your chosen values for \(a,b,c\) in place of the letters.

b) The definition of absolute value appears, specifically tailored to your chosen expression and your chosen relation symbol \((=, >, <, \geq, \leq)\). (built-in pause)

c) This definition is demonstrated visually on the number line by showing the possible coordinate positions of your expression \(ax+b\). (built-in pause)

d) This visual image is translated into a compound mathematical sentence to solve. (built-in pause)

e) Finally, the solution to the compound sentence appears and remains highlighted.

Note: If you wish to stop at any of the steps above, re-press the activated button (it will be darkened). To continue, the button must be restarted.

Alterations:

1. Before each new problem, always press the “Reset” button to clear the previous solution.
2. Press “Reset”; then choose a new sentence to solve from the 5 brown buttons at the left of the screen, or press “Reset” and first choose new values for \(a,b,c\) by double-clicking the values of \(a,b,c\) appearing in the upper left of the screen. For each one, a window opens up into which you can type your desired value. Press “OK” and your value is set.

Note: If \(c < 0\), there is no need to use algebra to solve the sentence. Since no absolute value is less than a negative, \(|ax + b| \leq c\) has no solution. Similarly, since every absolute value is greater than a negative, \(|ax + b| \geq c\) has all real numbers as solutions. These messages appear on the screen if this situation arises and a large red segment slashes through the work area. There also will appear specific directions on how to continue - follow them. Similarly, if \(c = 0\), most of these cases will not need a compound sentence, so the red slash and a quicker approach appears. Students always should be encouraged to “listen” to each problem first (repeat it out loud?) rather than be overly mechanical.
“Complex Rts of Parabola”

FINDING COMPLEX ROOTS OF A PARABOLA BY GRAPHING

We are very familiar with relating the real root(s) of a quadratic equation to the x-intercept(s) of the graph of its related function, a parabola. But what about the quadratic whose roots have an imaginary part and whose parabola does not intersect the x-axis? Did you know that we still can make a graphical connection? This animation explores that connection.

When first learning to solve a quadratic equation of the form \( ax^2 + bx + c = 0 \) (\( a \neq 0 \)), we generally have our students explore four possible techniques. Three of these techniques are algebraic – solving by factoring, solving by completing the square, and solving by the quadratic formula (this still is completing the square, but we jump to the final line saving ourselves many steps). However, prior to these more abstract solution techniques, it is advised that we first give our students a visual image of the situation, so we approach the problem graphically. We consider the graph of the related function \( y = ax^2 + bx + c \). This graph is, of course, a parabola. Here the values of the trinomial (the “y” values) are allowed to run the gamut beginning with the height of the parabola’s vertex and continuing unbounded in either a positive direction (if \( a > 0 \)) or a negative direction (if \( a < 0 \)). The only coordinates that solve our original equation, however, are the ones for which the trinomial equals 0 (i.e. \( y = 0 \)). Hence, the x-intercepts of the parabola (the graph of the related function) are the solutions to our original quadratic equation.

How to run this animation:

This animation opens with the initial parabola \( y = -\frac{1}{4}(x - 7)^2 - 4 \) showing 2 real roots at \( x=3 \) or \( x=11 \). Therefore, the solution to the quadratic equation \( 0 = -\frac{1}{4}(x - 7)^2 - 4 \) (which is the same as \( 0 = \frac{1}{4}x^2 - \frac{7}{2}x + \frac{33}{4} \)) is \( x=3 \) or \( x=11 \). Now drag vertex \( V(h,k) \) above the x-axis, possibly to \((7,1)\). As you do this, you will notice that the position of \( V \) automatically snaps to integer coordinates, for your convenience. (If you do not want this feature, simply go to the top pull-down menu “Graph” and release on “Snap Points” to deactivate. Keep in mind, however, that due to the discrete nature of a “pixelized” plane, you may not be able to land exactly on the coordinates you desire.) As you drag \( V \), you also will notice that in the upper left of the screen, the gray values of \( h \) and \( k \) are tracking the location of \( V \) and the blue equations are also adapting. Also at the left, the red “real r1” and “real r2” solutions change to “undefined” as soon as \( V \) travels above the x-axis and the pink “complex r1” and “complex r2” solutions take over. The parabola has now become \( y = \frac{1}{4}(x - 7)^2 + 1 \) with no real roots, but 2 complex roots at \( x = 7 \pm 2i \).

To see the graphical connection to these complex roots, access the 3 buttons in the gray box:

1. Plot Complex Roots & Connect … The complex roots are plotted using the x-axis as the real axis for the 7 and the y-axis as the imaginary axis for the \( \pm 2i \). (For additional information on plotting complex numbers along with some history, access the tab at the bottom of the screen labeled “history of complex nos.”, or run the animation in this packet called “Graph 3D or Complex Nos.”.)
2. Reflect Parabola over y=k … A brown reflected parabola appears along with its equation.
3. Rotate Complex Roots about Mdpt … The segment joining the plotted complex roots rotates 90° counterclockwise. The rotated complex roots land exactly on the x-intercepts of the reflected parabola.

Before continuing to another example, press “Reset” if you wish to clear the screen to the original graph. Anytime the parabola does not intersect the x-axis, you may run through the 3 buttons as above.

If \( r = 0.25 \) and \( V \) is dragged to \((6,2)\), the complex roots are \( x = 6 \pm 2i\sqrt{2} \), but will be rounded to \( x = 6 \pm 2.828 \) i.

Alterations:

The equation of the original parabola is driven by the coordinates \((h,k)\) of \( V \) (drag point \( V \) to alter them), and the value of \( r \) in the upper left of the screen. To alter \( r \), double-click this gray value. A window opens into which you can type your desired value. If \( r > 0 \), the parabola opens up, and if \( r < 0 \), it opens down.
How can we use this result?

With this result, we can tell the whole graphical story, not just part of it. We can present a complete analysis to the connection between the roots of a quadratic equation and the graph of its related quadratic function:

1. If the related parabola intersects the x-axis, those x-intercepts are the roots of the quadratic equation.
2. If the related parabola does not intersect the x-axis, the roots of the quadratic equation are complex numbers. To find them, reflect the parabola vertically over its vertex. Find the x-intercepts of that reflected parabola and rotate them 90 degrees about their midpoint. The coordinates of these rotated points, written as complex numbers of the form (d+ei) and (d-ei) using their x-coordinate as d and their y-coordinate as e, will be the desired roots.

Why does it work? (Proof)

Let’s compare the roots of a parabola that does intersect the x-axis with the roots of its vertical reflection over its vertex (this parabola will not intersect the x-axis):

Let \( y = ax^2 + bx + c \) represent any parabola that does intersect the x-axis.

By the quadratic formula, its real roots will be at \( \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \).

*Recall that the axis of symmetry of this parabola is \( x = -\frac{b}{2a} \) so we see its roots can always be found by starting at the intersection of the axis of symmetry with the x-axis and traveling left or right \( \frac{\sqrt{b^2 - 4ac}}{2a} \) units.

By completing the square, the equation of this parabola also is \( y = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \).

Hence its vertical reflection over its vertex is \( y = -a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \) which expands to \( y = -ax^2 - bx + c - \frac{b^2}{2a} \).

As stated earlier, this reflected parabola will not intersect the x-axis.

Its complex roots will be at \( \frac{b}{-2a} \pm \frac{\sqrt{b^2 - 4(-a)(c - \frac{b^2}{2a})}}{-2a} \) which is \( \frac{b}{-2a} \pm \frac{-\sqrt{-b^2 + 4ac}}{-2a} \).

Since \(-b^2 + 4ac\) must be negative (to produce non-real roots), these complex roots can be written as \( \frac{b}{-2a} \pm \frac{i\sqrt{b^2 - 4ac}}{2a} \). Since the sign of the term following the \( \pm \) sign is irrelevant, these complex roots can be written as \( \frac{-b}{2a} \pm \frac{i\sqrt{b^2 - 4ac}}{2a} \).

This means that disregarding the “i”, these roots are identical to the roots of the original parabola that did intersect the x-axis. By plotting these complex numbers using the real part, \( \frac{-b}{2a} \), on the x-axis, and the imaginary part, \( \pm \frac{\sqrt{b^2 - 4ac}}{2a} \), on the y-axis, we are using almost the same technique as in the * line above. The only difference is that instead of traveling left/right of center, as we need to do, we are traveling up/down from center. To compensate, we rotate these plotted complex points 90° about their midpoint. Now the vertical movement from center is once again a horizontal movement from center and the rotated imaginary numbers match the roots of the parabola’s reflection.
“Conics Defs & Graphs”

DEFINING & GRAPHING THE CONIC SECTIONS

10 separate animations can be accessed from the page tabs (lower left of GSP screen):

- overview
- circle
- parabola 1
- parabola 2
- ellipse 1
- ellipse 2
- hyperbola 1
- hyperbola 2
- general form eq
- eccentricity

These 10 animations thoroughly explore the definitions, graphs, and equations of the four conic sections along with their eccentricity. Directions and suggestions for the use of each of these animations follow:

“How to run this animation:

Press the yellow “Animate Points” button in the center of the screen. The four conic sections (circle, parabola, ellipse, and hyperbola) are traced out from their definitions. The traces slowly fade and are retraced over and over. Run this animation as you introduce the topic of conic sections prior to studying them one at a time. Have it run in the background as you discuss how the conic sections are intersections of a cone and a plane. Students may be curious about what they are seeing – especially about the pencil – and ask questions. You may choose to give brief explanations (see more details in the directions that follow for the subsequent animations) or assure them that all will be revealed in the days to come. Either way, relish in the interest and anticipation displayed by your students.

In the use of this overview animation as an introductory tool to the topic of conics, it is not recommended to use its featured adjustments (dragging points or lines), only the “Animate Points” button. Each of these adjustments, and more, will be featured in the subsequent animations that focus on each conic section individually. However, it is recommended that this overview be revisited, after your in-depth study of each of the conic sections, to review and compare/contrast the four figures. At that time, the adjustments listed as “alterations” below should be used.

If you wish the traces to be erased faster than they will fade, use the ESC key on your keyboard. You will need to press ESC multiple times if points are moving or selected (highlighted) on your screen since first the motion is stopped, then each selection is deactivated, and finally the traces are erased.

Alterations:

Circle: A locus of points in a plane a fixed distance (radius r) from a fixed point (center C).
- Drag the turquoise “alter r” point if you wish to adjust the fixed distance (radius) of the circle.
- Drag point C if you wish to move the fixed point (center) of the circle.
- Without the “Animate Points” button activated (re-click to stop), point P can be dragged manually.

Parabola: A locus of points in a plane equidistant from a fixed point (focus F) and a fixed line (directrix).
- Drag point F to move the fixed point (focus).
- Drag the directrix line to move the fixed line (it will move parallel to its current position).
- Drag the point at the outer tip of either arrowhead of the directrix to tilt it.
- Without the “Animate Points” button activated (re-click to stop), point D can be dragged manually.

Ellipse: A locus of points in a plane whose distances from two fixed points (foci) add to a constant.
- Drag the lavender point “adjust” to alter the constant sum of lengths PF₁ and PF₂.
- Drag points F₁ and/or F₂ to move the fixed foci. (They need not remain horizontal to each other.)
- Without the “Animate Points” button activated (re-click to stop), point P can be dragged manually.

Hyperbola: A locus of points in a plane whose distances from two fixed points (foci) differ by a constant.
- Drag the pink point “adjust” to alter the constant difference of lengths PF₁ and PF₂.
- Drag points F₁ and/or F₂ to move the fixed foci. (They need not remain horizontal to each other.)
- Without the “Animate Points” button activated (re-click to stop), point P can be dragged manually.
"circle"

This animation states and demonstrates the definition of a circle by dynamically sweeping out its locus points, derives the general equation \((x - h)^2 + (y - k)^2 = r^2\), and graphs a dynamic circle (its center and/or radius can be dragged) while displaying its specific equation. This equation automatically adjusts for changes as they occur.

How to run this animation:

1. Press the red “Collect Locus” button to trace out a circle by its definition. Point P automatically traces out the circle. The traces slowly fade and are retraced over and over allowing changes to be made to the center and radius. Alternatively, point P can be dragged manually to accomplish the same result.
2. Press the red “Show Circle” button for a “fade-proof” graph of the circle.
3. Press the lavender “Derive General EQ” button to see text that generates the circle’s general equation, \((x - h)^2 + (y - k)^2 = r^2\), from its definition.
4. Press the gray “Show Graph” button to see the circle graphed on a coordinate plane. Initially, the circle will be centered at \((3,2)\) with radius 4. The coordinates of its center, the length of its radius and its specific equation also are displayed on the screen. All of this data is dynamic which means that it automatically adjusts as points are dragged about the screen. Also notice that a small button appears just to the right of the “Show Graph” button. It is labeled “Hide EQ”. Press it and the coordinates of the center, the value of the radius, and the equation of the circle are hidden. Notice that this button has toggled to “Show EQ”. Press it again and the data returns. By hiding the data, altering the circle (see “Alterations” below), making a conjecture, and then showing the data, students can quickly test their ability to discern the equations of circles.
5. At any time, press the black “Reset” button to restore the screen to its initial state.

Alterations:

1. To alter the radius, drag the point labeled “alter r” at the upper left of the screen.
2. To move the center, drag point C.
3. If you would like center C to have only integer coordinates, go to the top pull-down menu “Graph” and release on “Snap Points”. Dragging C now causes it to jump to points with integer coordinates, however \(r\) continues to offer a full variety of values.
"parabola 1"

This first of two animations devoted to defining and graphing parabolas, states and demonstrates the definition of a parabola by dynamically sweeping out its locus points, shows (step-by-step) the compass/straightedge construction of those locus points, and derives the general equation \( y = \frac{1}{4r}(x - h)^2 + k \).

How to run this animation:
1. Press the brown “See Construction” button. Each step of the construction process, there will be five in all, is announced and shown with a 5 second pause between steps (do not intervene between steps). [To increase or decrease the length of this pause: right-click the brown “See Construction” button, click on “Properties”, choose the “Presentation” tab, enter your desired time, press “OK”.] Also, a new blue button titled “Collect Locus (or drag D)” appears. Press it and point D sweeps across the directrix tracing out the locus. The traces slowly fade and are retraced over and over allowing changes to be made to the positions of the focus and directrix (see “Alterations” below). Alternatively, point D can be dragged manually to accomplish the same result.
2. Press the red “Show Parabola” button for a “fade-proof” graph of the parabola.
3. Press the lavender “Derive General EQ” button to see text that generates the parabola’s general equation, \( y = \frac{1}{4r}(x - h)^2 + k \), from its definition. (\( r \) is the distance from focus to vertex.)
4. At any time, press the black “Reset” button to restore the screen to its initial state.

Alterations:
1. To move the focus, drag point F. (Notice what occurs when F moves to the other side of the directrix.)
2. To move the directrix (parallel to its current position), drag the body of the black line.
3. To tilt the directrix, drag either of the points at the tips of both arrowheads on the directrix. Dragging these points can also alter the length of the drawn directrix.

"parabola 2"

This second of two animations devoted to defining and graphing parabolas, graphs a dynamic parabola (its focus and/or directrix can be dragged) while displaying its specific equation. This equation automatically adjusts for changes as they occur. It will be in the form \( y = \frac{1}{4r}(x - h)^2 + k \) or \( x = \frac{1}{4r}(y - k)^2 + h \).

How to run this animation:
1. Press one of the two blue buttons under the text “Choose Directrix:”. With a horizontal directrix, the parabola will open either up or down; and with a vertical directrix, it will open either left or right. In either case, the following data will appear: the value of \( r \) (distance from focus to vertex), coordinates of both the vertex and focus, and the equation of the directrix. All of this data is dynamic which means that it automatically adjusts as points are dragged about the screen.
2. Press the red “Show EQ” button. A dynamic version of the parabola’s equation appears. It will be in the form \( y = \frac{1}{4r}(x - h)^2 + k \) or \( x = \frac{1}{4r}(y - k)^2 + h \). Notice that the button’s label has toggled to “Hide EQ”. Press it again and the equation is hidden. By hiding the equation, altering the parabola (see “Alterations” below), making a conjecture, and then showing the equation, students can quickly test their ability to discern the equations of parabolas.
3. At any time, press the black “Reset” button to restore the screen to its initial state.

Alterations:
1. To move the vertex, drag the red point “Vertex \((h,k)\)”. It “jumps” to points with integer coordinates.
2. To move the directrix (it will remain horizontal or vertical), drag the body of the blue line. It also “jumps”.
3. To change the directrix from horizontal to vertical, or vice versa, select the appropriate blue button at the upper left of the screen under the text “Choose Directrix:”.
4. To deactivate the “jumping” feature, go to the top pull-down menu “Graph” and release on “Snap Points”.
This first of two animations devoted to defining and graphing ellipses, states and demonstrates the definition of an ellipse by dynamically sweeping out its locus points (simulating the classic pencil and string drawing), displays key vocabulary, and explores how the lengths of the axes relate to the position of the foci and the constant sum.

How to run this animation:
1. Press the red "Collect Locus" button. The pencil moves and text explains the simulation. Since the definition of an ellipse requires the sum of the distances from P to each of the fixed foci to remain constant, the length of the string (attached to the foci) is that constant sum. As the pencil moves, it keeps the string taut, and traces out the ellipse. The traces slowly fade and are retraced over and over allowing changes to be made to the positions of the foci and constant sum or string length (see “Alterations” below). Alternatively, point P can be dragged manually to accomplish the same result. The dynamic values of the distances from P to each of the foci, along with their sum, appear at the upper right of the screen.

2. Press the red “See Ellipse & Data” button. The motion stops (but can be resumed by pressing the “Sweep Pencil” button), and the following data appears: a “fade-proof” graph of the ellipse, its center, major and minor axes, and the dynamic values of a, b, and c (they adjust automatically as points are dragged about the screen) along with how they are determined.

3. Press the blue button labeled “Why must sum (string) = 2a?” Text appears that explains why the constant sum = 2a. Follow the directions telling you to drag point P to point “a”, but you also will want to lift P just barely off of point “a” to clarify the rest of the explanation in the text.

4. Press the green button labeled “Support Calculation of b”. P moves to point “b”. A right triangle appears with legs of length “b” and “c” (“c” is the defined distance from focus to center), and hypotenuse of length “a”. The green triangle is congruent to its reflection over the minor axis, so the hypotenuse of each is congruent; and since both of them add to the length of the string (2a), each one must have length “a”. The Pythagorean theorem then calculates “b”.

5. At any time, press the black “Reset” button to restore the screen to its initial state.

Alterations:
1. To move the foci, drag either of the points F₁ or F₂.
2. To alter the constant sum of the distances PF₁ + PF₂ (string length), drag the brown point labeled “adjust” at the end of the string segment just beneath the turquoise definition box. Note that if you make the constant sum (string length) too short or cause the foci to be too far apart, its length cannot stretch from focus to focus and the ellipse does not exist.

This second of two animations devoted to defining and graphing ellipses, derives the general equation
\[
\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,
\]
and graphs a dynamic ellipse (its center and axes’ lengths can be dragged) while displaying its specific equation. This equation automatically adjusts for changes as they occur.

How to run this animation:
1. Press the red “Show Specific EQ” button. The ellipse’s dynamic equation appears. It is in the form
\[
\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \quad \text{or} \quad \frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1.
\]
Notice that the button’s label has toggled to “Hide Specific EQ”. Press it again and the equation is hidden. By hiding the equation and then altering the ellipse (see “Alterations” below), students can quickly test their own ability to discern the equation.

2. Press the “Derive General EQ” button for text that generates the ellipse’s general equation from its definition.

3. At any time, press the black “Reset” button to restore the screen to its initial state.

Alterations:
1. To alter the lengths of the horizontal and vertical axes, drag either of the two large blue “control” points. Note that the major axis (the longer one) continues to be labeled with length 2a and the minor axis (the shorter one) with length 2b. The position and coordinates of the foci, as well as the values of “a,” “b,”, and “c” (in the upper right of the screen) also are dynamic and adjust automatically.

2. To move the center of the ellipse, drag the green point labeled “(h,k)”. The dynamic display of its coordinates appears in the upper right of the screen and a vector tracks its displacement.
"hyperbola 1"

This first of two animations devoted to defining and graphing hyperbolas, states and demonstrates the definition of a hyperbola by dynamically sweeping out its locus points, displays key vocabulary, and explores how the asymptotes, the transverse and conjugate axes, the vertices, the foci, and the constant difference are related.

How to run this animation:
1. Press the red “Collect Locus” button. A hyperbola is traced out by its definition. Since all radii of a circle are congruent, the brown circle shows that, at all times, distance \(PF_1\) is the same as the distance subtracted from \(PF_2\) leaving a constant difference (the thick blue segment). The traces slowly fade and are retraced over and over allowing changes to be made to the positions of the foci and constant difference (see “Alterations” below). Alternatively, point \(P\) can be dragged manually to accomplish the same result. The \(\textit{dynamic}\) values of the distances from \(P\) to each of the foci, along with their difference, appear at the upper right of the screen.
2. Press the red “Show Hyperbola?” button (toggles to a “Hide” button) for a “fade-proof” graph of the hyperbola.
3. Press the yellow “Show all Data” button. The following appears: center, transverse and conjugate axes, asymptotes, and \(\textit{dynamic}\) values for \(a, b,\) and \(c\) (they adjust automatically as points are dragged about the screen) along with how they are determined.
4. Press the blue “Why is \(2a = \text{diff.}\)" button. Text appears that explains why the constant difference = \(2a\).
5. Press the brown “Why is \(\text{diff.} < \text{dist. } F_1 \text{ to } F_2?\)" button. Text appears that explains why the constant difference must always be less than the distance \(F_1F_2\).
6. Press the green button labeled “Support Calculation of \(b\)”. A text box appears with 4 buttons in it. Follow the steps of the text and access the buttons as indicated to explain the computation of \(b\) and validate the asymptotes.
7. At any time, press the black “Reset” button to restore the screen to its initial state.

Alterations:
1. To move the foci, drag either of the points \(F_1\) or \(F_2\).
2. To alter the constant difference of the distances \(PF_1\) and \(PF_2\), drag the brown point labeled “adjust” at the end of the segment just beneath the turquoise definition box. Note that if the constant difference exceeds \(F_1F_2\), or if the difference is zero (causing \(PF_1 = PF_2\)), the hyperbola cannot exist as explained in the text revealed by the brown button labeled “Why is \(\text{diff.} < \text{dist. } F_1 \text{ to } F_2?\)".

"hyperbola 2"

This second of two animations devoted to defining and graphing hyperbolas, derives the general equation
\[
\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,
\]
and graphs a \(\textit{dynamic}\) hyperbola (its center and axes’ lengths can be dragged) while displaying its specific equation. This equation automatically adjusts for changes as they occur.

How to run this animation:
1. Press the red “Show Specific EQ” button. The hyperbola’s \(\textit{dynamic}\) equation appears. It is in the form
\[
\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad \text{or} \quad \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1.
\]
Notice that the button’s label has toggled to “Hide Specific EQ”. Press it again and the equation is hidden. By hiding the equation and then altering the hyperbola (see “Alterations” below), students can quickly test their own ability to discern the equation.
2. Press the “Derive General EQ” button for text that generates the hyperbola’s general equation from its definition.
3. At any time, press the black “Reset” button to restore the screen to its initial state.

Alterations:
1. To alter the length of the transverse axis, drag the large red point “\(a\)” either above or to the right of \((h,k)\). Note that the coordinates of the foci and vertices, and the values of “\(a, b,\)” and “\(c\)” (in the upper right of the screen) are \(\textit{dynamic}\) and adjust automatically as points are dragged about the screen.
2. To alter the length of the conjugate axis, drag the large blue point.
3. To move the center of the hyperbola, drag the green point labeled “\((h,k)\)”. The \(\textit{dynamic}\) display of its coordinates appears in the upper right of the screen and a vector tracks its displacement.
This animation uses the general form equation of all conics and, through the use of sliders for the coefficients of the equation, morphs the conic sections from one to another.

How to run this animation:
The opening equation and graph are that of the unit circle. To alter the values of coefficients A,B,C,D,E,F drag the red points (sliders) on the number lines in the lower left of the screen. Follow the directions in steps #1-5 of the text on the screen. Begin with the unit circle showing on the screen.

1. Press the gray “Reset to unit circle” button to restore the unit circle after each move. Drag each slider to find:
   - F controls the radius of the circle (Note: if F > 0, \( x^2 + y^2 < 0 \) which has no real solutions.)
   - A controls the horizontal axis of an ellipse (A>0) or hyperbola (A<0). If A=0, the conic splits into 2 horiz. lines.
   - C controls the vertical axis of an ellipse (C>0) or hyperbola (C<0). If C=0, the conic splits into 2 vert. lines.
   - B stretches the ellipse along an oblique line (y=x or y=-x) until it splits into 2 parallel lines, then into a hyperbola.
   - D moves the ellipse’s center horizontally, keeping the y-intercepts unchanged (since x=0, D has no effect there).
   - E moves the ellipse’s center vertically, keeping the x-intercepts unchanged (since y=0, E has no effect there).
2. & 3. Press the blue “Hide graph” button. Press the red “Set A=1…” button to create the equation \( x^2 - y = 0 \).
3. Make a conjecture about this graph then re-press the blue button that had toggled into a “Show graph” button.
4. As in step 1, restore the parabola after each move using the red button in step 3. Drag each slider to find:
   - F raises (F>0) or lowers (F<0) the parabola.
   - D moves the parabola’s vertex horizontally, keeping the y-intercept unchanged (since x=0, D has no effect there.)
   - E moves the parabola’s vertex vertically, keeping the x-intercepts unchanged (since y=0, E has no effect there).
   - B causes an oblique hyperbola. Equation becomes \( y = -\frac{x^2}{Bx-1} \) with asymptotes \( x = \frac{1}{B} \) & \( y = -\frac{x}{B} - \frac{1}{B^2} \).
Continue exploring the effects of slider changes with the graph showing or practice recognizing conics from their equation by: hiding the graph, altering the equation, conjecturing, then showing the graph.

Alterations:
To enter the coefficient values by typing (rather than sliding), follow the directions at the lower right of the screen.

There are three general conic sections – the ellipse, parabola, and hyperbola (the circle is a special case ellipse in which the 2 foci have merged into one point). All three of these conic sections can be mutually defined as the locus of points R in a plane such that the ratio of the distance from R to a fixed point, to the distance from R to a fixed line, is a positive constant e (eccentricity). Controlled by the value of e, this animation dramatically demonstrates this concept. When \( 0 < e < 1 \), the conic is an ellipse. When \( e = 1 \), it is a parabola. When \( e > 1 \), it is a hyperbola.

How to run this animation:
1. Press the red “Sweep e” button. Point e moves along a portion of a number line in the upper left of the screen, taking on values from 0 to approximately 2. The dynamic value of the eccentricity (e) is displayed beneath it. Under the value of e, is the dynamic ratio RF/RH that at all times is equal to e. The conic, which began as an ellipse, changes for an instant to a parabola (if you can catch it), then to a hyperbola and continues to cycle through these three shapes as the eccentricity changes. The text in the large gray box matches values of e to the resulting conic section.
2. Press the red “Move e -> 1” button to capture the parabola.
3. If point e is moving, notice that the “Sweep e” button is darkened. Stop e by re-pressing the button. Manually drag e to cause the conic section of your choice. Press the “See all pts. R” button to demonstrate that all locus points R satisfy the definition in the gray box. Point R moves about the plane gathering only those points whose ratio of distances from F and from the directrix are in the ratio e. Note that the dynamic ratio RF/RH shows RF and RH changing value but that their ratio remains constant and equal to e. Change e and test out a different conic section.

Alterations:
1. Alter the eccentricity by dragging the red point “e” on the short number line at the upper left of the screen.
2. Alter the position of the fixed point or fixed line by dragging point F or the body of the directrix.
“Conics Interesting Properties”

MISCELLANEOUS PROPERTIES & APPLICATIONS OF CONICS FOR ENRICHMENT

5 separate animations can be accessed from the page tabs (lower left of GSP screen):

- “oids’ photos
- hyperbola family
- mutually orthogonal
- reflections & collections
- falling bodies

The conic sections are a most fascinating group of shapes, studied for thousands of years. They are so rich in interesting features and real-world applications that this is a perfect time to address the age-old question, “What is this ever going to be used for?” These animations present just a few of these features and applications. Use them as enrichment with your class, or as a springboard for further independent study.

“oids’ photos”

A parabola, ellipse, and hyperbola, when revolved about their respective axes of symmetry, become 3-dimensional shapes called a paraboloid, ellipsoid, and hyperboloid. Just as a circle is a special ellipse, so is a sphere a special ellipsoid. This first animation of the bundle demonstrates how these shapes are formed and provides a few examples in which nature, architects, and engineers have used them.

How to run this animation:
Simply press the black “Spin” button at the center bottom of the screen. A parabola, ellipse, and hyperbola spin, sweeping out a 3-dimensional paraboloid, ellipsoid, and hyperboloid. The two photos on the left show satellite dishes (paraboloids). A paraboloid reflects all parallel sound or light waves to its focus (tip of the central “pyramid”), where it is amplified and collected (see “reflections & collections” animation for more). The three center photos feature ellipsoids. Consider not only the planet shapes themselves, but also their elliptical paths (Johannes Kepler’s first law of planetary motion – “the orbits of the planets are ellipses, with the Sun at one focus of the ellipse”). The three photos on the right show architecture inspired by the hyperboloid.

Press the black “Reset” button to restore the screen to its initial state.

“hyperbola family”

How to run this animation:
Press button #1 followed by button #2. The pink concentric circles will overlay onto the blue ones and then slide to the right revealing patterns of interference forming a family of hyperbolas overlaying a family of ellipses as shown at the right of this page ⇒ The “families” are confocal (same foci, the two centers) with varying eccentricities. Each hyperbola and ellipse intersect at right angles. This means that they are “mutually orthogonal” (see next animation for more). Ah, but is anything really there besides the circles?

Press the black “Reset” button to restore the screen to its initial state. The next animation explores this idea further.

“mutually orthogonal”

After running the previous “hyperbola family” animation, this one provides a closer look at confocal (shared foci) ellipses and hyperbolas being “mutually orthogonal”. It also shows two parabolas that are mutually orthogonal.

How to run this animation:
Follow steps #1-4 for both situations, pressing the buttons and dragging points when indicated. Those instructions should be self-explanatory. The appearance of the tangent segments provides an excellent opportunity to introduce the concept of “local linearity” to your students and lay an important building block for calculus. Basically, these curves have “local linearity” because in a tiny interval around any point P, the curve is essentially “straight”. The tangent segment at P is an extension of that linear behavior around P. (In contrast, the graph of y = |x|, for instance, does not have local linearity at the origin because no matter how small an interval you choose around the origin, the jagged turn always is present.)
“reflections & collections”

Parabolas and ellipses have powerful reflective properties that allow amplification and collection of light and sound waves that are used in car headlights, flashlights, satellite dishes, band shells, “whisper chambers”, etc. These properties are demonstrated and explained in this animation. (For fun, an elliptical pool table also is included.)

How to run this animation: (Be sure to press the “Reset” button between each of the 3 demonstrations.
Under “Paraboloids” …
1. Press button #1. A single particle is launched from the paraboloid’s focus. It reflects off the surface following the rules of physics – the two red angles must be congruent (see text box that appears on the screen for more detail). Conversely, when an exterior particle hits the paraboloid, it is reflected to the focus.
2. Press button #2 (or manually drag point P). Identical results for all points of impact are seen.
3. Press button #3. Many radiating particles from the focus all hit the paraboloid at various times and places but, due to the precise shape of the paraboloid, once reflected, their parallel paths form a uniform beam. Conversely, exterior parallel particles will collect at the focus simultaneously (collection/amplification).
Under “Elliptical Pool Table” … (Just one pocket at one focus point exists. The objective is to “bank” the ball into it.)
1. Press button #1 or choose the “Random” button (wait for 4th position) to place the ball on the table.
2. Press button #2. A cue stick hits the ball. It reflects off the ellipse wall directly into the pocket at F2.
3. Press button #3. An additional demonstration and explanation appear on the screen. Be sure to revisit the definition of an ellipse (and sliding pencil) in the “ellipse 1” animation in the “Conics Defs & Graphs” file.
4. Press the “Animate P” button (or manually drag point P). P slides to show that any ball hit so that it crosses F1, will reflect off the ellipse and land in the pocket at F2 (assuming a “clean” hit and sufficient velocity).
Under “Elliptical Room – Whisper Chamber” (Ask if any of your students has experienced this type of room at the U.S. Capitol, London’s St. Paul’s Cathedral, or Chicago’s Museum of Science & Industry)
1. Press the “Why?” button. Sounds radiate in several directions from F1, reflect off the ellipse, and simultaneously collect at F2. An explanation appears as the simulation reverses and repeats.
2. Press the “Animate P” button (or manually drag point P). P slides to show that sounds in all directions, emanating from F1, will reflect and collect simultaneously at F2. Hence, the whisper is greatly amplified.

Note: The tangent segments in these demonstrations provide an opportunity to discuss the concept of “local linearity”. (See the explanation in this booklets instructions for the previous animation called “mutually orthogonal”.)

“falling bodies”

Gravity (acceleration) on Earth pulls at 9.8 m/sec^2, causing all falling paths to be parabolas. These paths are easily observed when throwing/hitting/kicking balls in all sports; firing a arrow or a slingshot; diving off a diving board (especially the cliff divers in Acapulco); in water streaming from a garden hose or sprinkler; in waterfalls, etc. This animation demonstrates how a projectile falls off of its initial path due to gravity. The user chooses the projectile’s launch angle, initial velocity, and the duration of the event viewed. There also is a bit of target practice. The projectile’s coordinates are: x = (initial velocity) (time) [cos (launch angle)]
y = (-1/2 gravity) (t^2) + (initial velocity) (time) [sin (launch angle)] + (initial height)

How to run this animation:
1. Press the brown “Shoot” button. Using the preset opening launch angle of 50°, initial velocity of 20 units/sec, and a domain for t that matches the length of time it will take the projectile to hit the ground, the projectile appears and leaves a trail. It’s path without gravity is dashed and the displacement due to gravity is drawn. Its dynamic coordinates and its displacement due to gravity appear in the upper right of the screen. To stop the animation, re-press the darkened “Shoot” button. To pause or slow it down, use the top pull-down menu “Display” and release on “Show Motion Controller”. Alternatively, the pink point t can be dragged manually.
2. Press the red “Try to hit a target?” button. Press the brown “Shoot” button again. If the target is missed, a change in velocity or launch angle is needed (see “Alterations” below). Press the “Reset” button before making the changes if desired. A large message will appear on the screen when the target is hit.
3. Pressing the “Reset” button does not reset the angle and velocity values or the scale of the axes (if changed).

Alterations:
1. To alter launch angle, double-click the blue angle measure. A window opens. Type in the desired value.
2. To alter initial velocity, double-click the green velocity measure. A window opens. Type in the desired value.
3. To alter the domain of t, drag the pink “max t” point, or press the pink “Move max t -> hit ground” button. (This automatically ends the domain when the projectile hits the ground.)
4. To re-scale the axes, drag the red point at (1,0).
“Coordinate Plane Basics”

BASIC VOCABULARY AND GRAPHING POINTS ON THE CARTESIAN PLANE

This simple animation introduces basic vocabulary of the coordinate plane, and dynamically connects points on the plane to their ordered pairs. It also provides a bit of history about the 17th century mathematician, René Descartes.

How to run this animation:

1. Press each of the first 3 blue buttons in the upper left of the screen. They indicate the axes and origin. Note that the last of these buttons toggles between “Hide” and “Show” each time it is pressed.
2. Press each of the 4 “Show Quadrant” buttons. They toggle between “Hide” and “Show” each time pressed to allow mixing and matching of highlighted quadrants. To hide all 4 quadrant highlights, press the “Hide All Quadrants” button. It does not toggle.
3. Drag points A, B, C and note that their coordinates are displayed dynamically (adjusting as the points are moved about the screen). Point A can move freely anywhere. Points B and C each remain on one axis to emphasize that axis points have one coordinate that is 0 constantly. To practice naming coordinates, press the “Hide Coordinates” button beneath A, B, or C (note that it toggles to a “Show” button). Drag the point, conjecture its coordinates, and press the “Show Coordinates” button to confirm.
4. If you want point A to have only integer coordinates, go to the top pull-down menu “Graph” and release on “Snap Points”. When dragged, it will then jump to these spots, however do emphasize to your students that points are not really limited to having only integer coordinates. Points B and C will continue to offer a full variety of values for their non-zero coordinates.

Press the “Rene DesCartes” tab (bottom of the screen) for a bit of information on the “father of analytic geometry”. (The name “Cartesian” plane, the alternate name for the coordinate plane, was given in his honor.) Be sure to scroll down to the bottom of the text using the scroll bar at the right of the screen.
“Cycloids, Sq Wheels, Ferris”

BUILDING GRAPHING SENSE WITH 3 INTERESTING SITUATIONS

3 separate animations can be accessed from the page tabs (lower left of GSP screen):

- **cycloids**
- **square wheels**
- **Ferris wheel**

These 3 animations build basic graphing sense, suitable for pre-algebra, using a pebble stuck in a wheel, a square-wheeled cart, and a Ferris wheel. However, since they reveal advanced curves (cycloid, sine), they also are suitable through pre-calculus. In addition, they are great to explore on parent nights - engaging, non-threatening, and fun!

**How to run this animation:**

1. Speculate the answer to the “What if” question in red text on the screen (groups are great here). Have students commit to a response by drawing it. (Have them share and defend their drawings?)
2. To test the conjectures press the “3. Roll” button (initially, no alterations are needed). The wheel rolls and the pebble’s path is traced out. This shape is a “cycloid”. For additional rolls, first press button #1.

   In 1696 Johann Bernoulli published a problem in Leibniz’s journal *Acta Eruditorum*. The challenge: Consider point A higher than B but not directly over B. What is the shape of the curve down which a particle must slide (without friction) under the influence of gravity so as to pass from A to B in the least time? He called this curve the “brachistochrone” from the Greek words for “shortest” and “time”. Newton’s solution was a cycloid.
3. To explore epicycloids and hypocycloids: first ask your students what the same pebble’s path would be if the wheel rolled around a circle instead. (A very perceptive student might ask if the wheel is inside or outside of the circular path.) Press the “Show” button at the top of the screen (note that it toggles into a “Hide” button). Speculate the answer to the question posed. When ready to test the conjectures, press the “Roll for Epi/Hypo” button. The curved cycloids appear (ignore the original wheel spinning in place). What would cause the paths to end exactly where they began? (answer: large radius divisible by small radius)

4. At any time, to erase traces, press the ESC key on your keyboard (possibly more than once).

**Alterations:**

1. To resize the wheel, drag the red point W at the top left of the screen. Then, choose the “rev.” button (there are 6 of them) that matches the largest number of wheel revolutions that will fit on your screen.
2. To resize the pebble, drag the red point P at the top left of the screen.
3. To resize the circular path for the Epicycloid/Hypocycloid, drag the red point under “Adjust rim size”.

**“square wheels”**

**How to run this animation:**

1. Speculate the answer to the “What if” question in red text on the screen (groups are great here). Explain that the center (axle) must stay the same distance above the ground to produce the “smooth ride”. Have students commit to a response by drawing their road. (Have them share and defend their drawings?)
2. To give a hint, press the “Roll Wheel” button. The wheel rolls, but the road is not revealed. Re-conjecture.
3. To test the conjectures, first put a trace on the interior of the square wheel: stop the wheel (re-press the darkened “Roll” button, or press “Reset”); click on the blue interior to highlight it; from the drop-down menu “Display” at the top of the screen, release on “Trace Quadrilateral”. Again press the “Roll Wheel” button. The road appears.

**Alterations:**

To resize the wheel, drag the red “adjust” point at the lower right of the screen. Press “Reset”. Press “Roll Wheel”.

**“Ferris wheel”**

**How to run this animation:**

1. Press the “Rotate Ferris Wheel” button. The wheel spins. (May also manually drag the large red point.)
2. Have students draw a response to the problem posed in the red text at the top of the screen.
3. Press the “Create Graph” button. A shifted sine wave develops. The red length (x-coordinate) matches the swept arc around the wheel, and the green length (y-coordinate) matches the height of the yellow chair.
4. At any time, press the “Reset” button to restore the screen to its initial state.
“Graph 3D or Complex Nos”

GRAPHING IN XYZ-SPACE OR ON THE COMPLEX NUMBER PLANE

2 separate animations and a history page can be accessed from the tabs (lower left of GSP screen):

- [graph points in 3D]
- [graph complex numbers]
- [history of complex numbers]

“graph points in 3D”

This animation drops the xy-plane horizontal, adds a vertical z-axis, and demonstrates how to graph a point in space.

How to run this animation:
1. Press the “Drop the y-axis flat” button. The xy-plane is now horizontal and the z-axis is vertical.
2. Double-click each of “new x”, “new y”, “new z” and type desired values into the windows that appear.
3. Press the “Plot (x,y,z)” button. From the origin, a point travels according to its 3 coordinates. Also showing are trails of its paths, its coordinate positions on the 3 axes, and a rectangular solid to help visualize this point in space. A large *dynamic* ordered triple for this point appears in the lower right of the screen.
4. Coordinate points x, y, and z may be dragged manually instead. Press the “Hide Coords.” button first (toggles to a “Show” button) to have students discern the ordered triple. Press the “Show” button to confirm.
5. At any time, press the “Reset” button to restore the screen to its initial state.

To practice: Select values for x,y,z according to step #2 above. Drag the gray point to where it is conjectured that the red point will be, then press button #3 to check the result.

“graph complex numbers”

This animation graphs a complex number, a+bi, using the x-axis for the real part (a) and the y-axis for the imaginary part (bi). An additional page tab accesses a history page chronicling the development of complex numbers.

How to run this animation:
1. Manually drag point P around the plane. A *dynamic* ordered pair adjusts to its change of position.
2. To type the coordinates for point P, press the “1. How to set values for a & b” button and follow the directions that appear on the screen. Then, press the “2. Graph (a+bi)” button.
3. At any time, press the “Reset” button to set P to the origin.
“Graph Classic Functions”

7 separate animations can be accessed from the page tabs (lower left of GSP screen):

| poly | trig | exp | log | parametric | polar | your choice |

These powerful animations use sliders (for dynamic coefficients that “morph” the graph), to develop a deep and meaningful understanding of the behavior of classic functions and transformations, such as shifts and stretches.

This animation demonstrates the evolution of a polynomial from a constant function through a quintic function.

How to run this animation:
1. Press the “Hide general…” button (toggles to a “Show” button) if it is inappropriate for your particular class.
2. Press as many of the 6 blue buttons as desired. Each begins with the previous polynomial (one degree less) and shows how the addition of each higher power of x adds one more possible direction to the curve. (Also note that the sign of the leading coefficient determines the right-hand end behavior of the curve (positive -> up, negative -> down). Appropriate vocabulary, a dynamic equation specific to this degree function, and suggested changes for sliders also appear on the screen. Drag the sliders to explore these changes.
3. Press the “Slide x” button. A dynamic point travels the pink domain of the curve (or drag point x manually).
4. At any time, press the “Reset” button to restore the screen to its initial state.

Other than the leading coefficient effect (see above), the effects of the sliders are:
- k: vertical translation (slide)
- e: slope at x=h
- h: horizontal translation (slide)
- v: vertical dilation (stretch/shrink)

Alterations to domain:
Drag points M and/or N to alter the domain (viewing window) of the curve. The rest of the curve is pale and dashed.

How to run this animation:
1. The first 3 buttons toggle between “Show” and “Hide”. Use them to choose graphs of sine, cosine, and/or tangent. Drag slider n to -1 (this will be an exponent, not notation for arcsin, etc.) to see cosecant, secant and/or cotangent. Initially, only sine is shown.
2. Press the “Slide x” button. A dynamic point travels the pink domain of the curve (or drag point x manually).
3. Drag the sliders to explore changes. The slider effects are.. a: amplitude, b: period, c & d: horiz./vert. translation (shift), n: power of sine, m: power of (x-c).
4. At any time, press the green “Sliders to y=sin x, y=cos x, y=tan x” button to set the basic parent curves.
5. At any time, press the gray “Reset” button to restore the screen to its initial state.

Alterations to domain:
Drag points G and/or H to alter the domain (viewing window) of the curve. The rest of the curve is pale and dashed.

These two animations explore exponential and logarithmic functions. Dynamic function expressions track changes.

How to run these animations:
1. Press the “Slide x” button. A dynamic point travels the pink domain of the curve (or drag point x manually).
2. Drag the sliders to explore changes (changes to base, b, are most interesting). The slider effects are.. d & k: horiz./vert. translation (shift), a & c: vert./horiz. dilation, b: base, n: power of (x-d) or power of log.
3. Drag either “Set base..” button for the base mentioned on the button.
4. At any time, press the green “Reset sliders..” button to set the basic parent curves.
5. On the “log” page, press the brown “Show Inverse” button (toggles to a “Hide” button) to discover its inverse.

Alterations to domain:
Drag points G and/or H to alter the domain (viewing window) of the curve. The rest of the curve is pale and dashed.
This animation allows a thorough dynamic exploration of parametric graphing. Any \( x(t) \) & \( y(t) \) may be used.

How to run this animation:
1. Press the red “Slide t” button. A dynamic point travels the pink domain (or drag point \( t \) manually). The dynamic values of \( t \), \( x(t) \), and \( y(t) \) appear in the upper left of the screen, adjusting to changes in \( t \). The equations for \( x(t) \) and \( y(t) \) appear in the lower right of the screen, but are labeled as \( X(x) \) and \( Y(x) \) due to limitations of labeling in GSP. Stress that the domain value used actually is \( t \), not \( x \)!
2. Press the blue “Show Graph” button. The graph of \( (x(t), y(t)) \) appears. Initially, this is a “Lissajous” curve.
3. Drag the sliders \( a, b, c, d, e, f \) to explore changes.

Alterations to domain, \( x(t) \), and \( y(t) \):
1. Drag points \( t_{\text{min}} \) and/or \( t_{\text{max}} \) to alter the domain of \( t \), or press any of the 3 pink buttons at the upper right.
2. To edit the \( x(t) \) and \( y(t) \) expressions, double-click the \( X(x) \) and/or \( Y(x) \) equations in the lower right of the screen. A window opens. Type any desired function. Values pi and e are accessible under the “Values” bar of this window. Many functions (sine, etc.) are accessed from the “Functions” bar. For \( t \), click on “x” in the window. Although it is mislabeled, it will use the \( t \) value in the animation. To imbed sliders, if desired, click on any of the values \( (a =, b =, ..., f =) \) above the number lines instead of typing a specific number when editing. When finished, click “OK”. The window closes; the edit is accepted.

This animation allows a thorough dynamic exploration of polar graphing with a complete display of all relevant data.

How to run this animation:
1. Press the red “Slide theta” button (or drag point \( \theta \) manually). Point \( P \) is graphed showing it always is a dilation \( (r) \) of the unit vector (green arrow) whose direction is controlled by \( \theta \). The dynamic values of \( \theta \) and \( r(\theta) \) appear in the upper left, adjusting to changes in \( \theta \). Traces of the graph slowly fade.
2. To move \( \theta \) to the origin, press the red “theta -> 0” button. Press either “Zoom” button to zoom.
3. Press the blue “Show r graph” button (toggles to a “Hide” button) for a “fade-proof” graph.
4. Stop the motion (re-press the darkened “Slide theta” button) and drag sliders \( a, b, \) and/or \( c \), to explore their effects on the graph.
5. To explore intersections of polar curves, press the “Show 2nd graph” button (toggles to a “Hide” button). Remember that it is insufficient for the paths of \( P \) and \( K \) to intersect - points \( P \) and \( K \) must coincide.

Alterations to domain, \( x(t) \), and \( y(t) \):
1. To alter the domain of \( \theta \), drag the endpoints (unlabeled) of the pink domain segment, or press any of the 4 pink buttons at the lower right of the screen.
2. To edit the \( r(\theta) \) expression, double-click it (just under the yellow title box). A window opens. Type any desired function. Values pi and e are accessible under the “Values” bar of this window. Many functions (sine, etc.) are accessed from the “Functions” bar. For \( \theta \), click on \( \theta \). The equation “\( r = .. \) (3rd blue line) will no longer apply to your edited function, but all else will be correct. To imbed sliders, if desired, click on any of the values \( (a =, b =, c =) \) just above the number lines on the left of the screen instead of typing a specific number when editing. When finished, click “OK”. The window closes; the edit is accepted.

Use this animation to create your own dynamic graph of any function or composite of functions desired.

How to run this animation:
1. Graph up to 3 functions of your choice using the “Hide graph” buttons (they toggle into “Show” buttons).
2. To edit any function, double-click the first of any of the paired data lines. A window opens. Type any desired function. Values pi and e are accessible from the “Values” bar of this window. Many standard functions (sine, etc.) are accessed from the “Functions” bar. For \( x \), click on \( x \). When finished, click “OK”.
3. For a specific function value, double-click the second of the paired data lines. A window opens for the edit.
4. Press the red “Slide x” button. A dynamic point travels the pink domain (or drag point \( x \) manually).

Alterations to domain: Drag points \( M \) and/or \( N \). The rest of the curve is pale and dashed.
“Greatest Integer”

GRAPHING GREATEST INTEGER FUNCTIONS

2 separate animations can be accessed from the page tabs (lower left of GSP screen):

- linear G.I. with open circles
- any G.I. but w/o open circles

“linear G.I. with open circles”

This animation explores graphs of the form \( y = c\lfloor ax + b \rfloor + d \) where \( \lfloor x \rfloor \) denotes the “greatest integer” function.

How to run this animation:
1. Using the initial values of the sliders, the opening graph is \( y = \lfloor x \rfloor \). Each interval is shown to be closed on the left and open on the right. The dynamic (blue) function expression adjusts as values change.
2. Drag the red slider points \( a, b, c, d \) to see how they affect the graph. \( b \) & \( d \): horizontal & vertical translation (shift), \( a \) & \( c \): horizontal & vertical dilation (stretch/shrink) Notice that the open circles (excluded endpoint) shift to the left side of each interval when \( a < 0 \).
3. At any time, press the red “Reset sliders to \( y = \lfloor x \rfloor \)” button to restore the graph to \( y = \lfloor x \rfloor \).

“any G.I. but w/o open circles”

This animation opens with \( y = \lfloor x^2 \rfloor \) but can be modified to any function with greatest integer. \( G(x) \) denotes \( \lfloor x \rfloor \), the “greatest integer” function.

How to run this animation:
1. Double-click the blue expression for \( f(x) \) beneath the yellow title box. A window opens. Edit the function typing anything around \( G(\ ) \) and anything inside of it. Values \( \pi \) and \( e \) are accessible from the “Values” bar of this window. Many standard functions (sine, etc.) are accessed from the “Functions” bar. For \( x \), click on \( x \). When finished, click “OK”. \( G(\ ) \) will execute the greatest integer function to whatever is inside the parentheses.
2. Although the graph of \( f(x) \) in this animation does not show which end of the intervals is included or excluded, it can be determined using a target value at the lower left of the screen. Double-click “target”. A window opens. Type an endpoint of an interval. Click “OK”. Press the “Move \( x \) -> target” button. \( x \) moves to your target value to show which interval it is on. This determines which endpoint is included and which is excluded. Also, the value of \( f(\text{your target}) \) appears in the bottom corner.
3. Point \( x \) may be dragged manually.

Suggestions:
- \( f(x) = \sin(20 \ G(5x) ) \)
- \( f(x) = e^{0.5 \ G(x)} \)
- \( f(x) = x + G(x) \)
- \( f(x) = \frac{x}{G(x)} \)
- \( f(x) = \frac{\lfloor x \rfloor}{G(x)} \)
- \( f(x) = \frac{G(x)}{x} \)
"Inverse of a Function"

GRAPH ANY FUNCTION ALONG WITH ITS INVERSE AND THE LINE Y=X

This animation originally begins with a polynomial for f but it may be changed into any other desired function. Not only does it graph f, its inverse, and the line y=x, but it also emphasizes the swap of domain and range and the fact that y=x is the perpendicular bisector of the segment joining any point of f to its image point in the inverse.

How to run this animation:
1. Press the “Show line y = x” button (toggles to a “Hide” button). The dashed line appears on the screen.
2. Press the “Show Inverse of P” button (toggles to a “Hide” button). The reflection of P across the line y=x appears. The dynamic values f(x) = y and f_inv(y) = x appear on the screen to emphasize the swap of domain and range values. The segment joining P to P_inv has the line y = x as its perpendicular bisector.
3. Drag the pink point x manually.
4. Press the “Show Inverse of f” button (toggles to a “Hide” button). The inverse appears with its range highlighted to emphasize that the range of the inverse is the domain of the original function.
5. Press the “Slide x on domain AB” button to see that the relationships mentioned above are maintained.
6. Drag the red slider points to “morph” f and its inverse. Mix and match what is shown, as desired.
7. At any time, press the “Reset” button to restore the screen to its initial state (except for function f, if edited).

Alterations:
1. Drag points A and/or B to alter the domain (viewing window) of f. The rest of the curve is pale and dashed.
2. To edit f(x) into any other function, double-click the f(x) expression beneath the yellow title box. A window opens. Type any desired function. Values pi and e are accessible under the “Values” bar of this window. Many functions (sine, etc.) are accessed from the “Functions” bar. For “x”, click on “x” in the window. To imbed sliders, if desired, click on any of the values (a =, b =, …, e =) in the lower left of the screen instead of typing a specific number when editing. When finished, click “OK”. The window closes; the edit is accepted.

Suggestion: This animation is excellent to use when introducing inverse trig. functions (arcsin x, arcos x, etc.) and learning what domain restrictions are necessary for their inverses to be functions. Edit f(x) to sin(x), or cos(x), etc.
"Linear Equations, Mdpt, Dist"

EXPLORE SLOPE, GRAPH LINES, DEVELOP MIDPOINT & DISTANCE FORMULAS

8 separate animations can be accessed from the page tabs (lower left of GSP screen):

- slope def.
- $y = mx + b$
- $Ax + By = C$
- system of 2 lines
- linear inequalities
- parallel & perpendicular
- mdpt formula
- dist formula

This cluster of animations addresses the basic concepts of graphing various forms of linear equations and inequalities, explores the concept of slope, solves systems, and develops the formulas for midpoint and distance.

"slope def."

This animation visually builds the essential concept of slope as a comparative ratio that can be found with or without numerical data.

How to run this animation:

1. Press the green "See Rise/Run (with measures)" button. $P_1$, $P_2$ and their dynamic coordinates appear. The physical rise and run between the points is created, and the value of the slope is calculated. A second green button also appears. Press it. $P_1$ and $P_2$ slide along the line. The dynamic calculations for $\Delta y$ and $\Delta x$ adjust to changes in $P_1$ and $P_2$ but the slope quotient remains constant.
2. Stop the motion (re-press any darkened buttons). Press either the "Tilt Horizontal" or "Tilt Vertical" to see $\Delta y$ or $\Delta x$ become 0 as the line moves.
3. Press the red "Show Equation of Line" button (toggles into a "Hide" button). Drag A and/or B to alter the line and explore new slopes. (Might some students notice that the slope is always in the line’s equation?)
4. Press the "Erase All But Line AB" button to reset the screen (doesn’t move the line) to its initial state.
5. Since slope is a ratio (definition appears at top of screen), its value can be approximated with surprising accuracy without using any measurements at all – just comparing the visual lengths of vertical rise and the horizontal run. To do this, follow the brown text directions on the screen. The "Hide Axes" button toggles to a "Show" button, but keep the axes hidden while performing this exercise. After step #3, allow students to modify their estimate. Press the toggled "Hide Slope Value" button and loop back to step #2 to repeat.
6. At any time, press the black "Reset" button to restore the screen to its initial state.

Rounded decimals can be misleading: When measuring, GSP uses a large number of digits but rounds off the displayed values. This may at times cause a calculation to appear to be off by one unit in the last digit. For example, if $2.3 + 1.4 = 3.6$ appears, the numbers may have been $2.26….. + 1.37…..$ which is $3.63…..$

"$y = mx + b$"

This animation provides practice and experience graphing equations of the form $y = mx + b$.

How to run this animation:

Follow steps #1-4 on the screen:

1. Resets the screen for a new example.
2. To enter new values for $m$ and $b$, double-click "m = " or "b = " in quadrant 2. A window opens. Enter the desired value and press "OK". (Values for $m$ may be entered as fractions.) Alternatively, click any of the equations only once and use the "+" or "−" keys on the keyboard to increase or decrease it. The dynamic equation in quadrant 3 reflects these changes.
3. Points $P_1$ and $P_2$ will snap to integer coordinates only. To disable this feature, from the top drop-down menu "Graph", release on "Snap Points" to deactivate it. (However, due to the discrete nature of a "pixelized" plane, it is difficult to land on specific desired coordinates.) When moving $P_1$ and $P_2$, it might be suggested to the students that they use one of them to place the y-intercept first.
4. A line appears and moves into the correct position. The equation of the student’s line and the correct equation appear together at the bottom of the screen for comparison. If the student’s answer is correct, a rewarding message is displayed. Loop back to step #1 to repeat.
This animation provides practice and experience graphing standard form linear equations $Ax + By = C$.

How to run this animation:
Follow steps #1-4 on the screen:
1. Resets the screen for a new example.
2. To enter new values for $A$, $B$, $C$, double-click "A = ", "B = ", or C = " in quadrant 2. A window opens. Enter the desired value and press "OK". Alternatively, click any of the equations only once and use the "+" or "−" keys on the keyboard to increase or decrease it. It is advised to have $C$ divisible by $B$ for beginning examples; if not, set $P_1$ and $P_2$ so the slope is correct, then drag the body of the line (parallel) until the y-intercept is correct. The dynamic equation in quadrant 3 reflects these changes.
3. Points $P_1$ and $P_2$ will snap to integer coordinates only. To disable this snapping feature, from the top drop-down menu "Graph", release on "Snap Points" to deactivate it (however, the pixel restrictions may be frustrating). When moving $P_1$ and $P_2$, it might be suggested to the students that they use one of them to place the y-intercept first.
4. ONLY ONE OF THE 2 RED BUTTONS APPLIES TO ANY ONE SITUATION. Press the one that is being pointed to. A separate button is needed for vertical lines because they are not functions and are handled differently by GSP. A line appears and moves into the correct position. The equation of the student’s line and the correct equation appear together at the bottom of the screen for comparison. If the student’s answer is correct, a rewarding message is displayed. Loop back to step #1 to repeat.
5. The tiny brown button at the lower right corner of the screen toggles between “Hide” and “Show” and asks the student to notice that in $Ax + By = C$, the slope is $−A/B$ and the y-intercept is $C/B$.
This animation explores the special slope relationships of parallel lines and perpendicular lines.

How to run this animation:
1. Press the “Create Parallel” button. A moving parallel line separates from \( \overline{AB} \). Drag A, B, and line h.
2. Press the “Create Perpendicular” button. A moving line rotates out of \( \overline{AB} \) and becomes perpendicular to it. There is a lot of dynamic data to watch here. Press the button a few times or use the Motion Controller (from the top pull-down menu “Display”, release on “Show Motion Controller”, drag to convenient place on screen) to pause the action to see all the dynamic (changing) data during the action: rotation angle, coordinates of \( Q \) becoming \((-y,x)\) as compared to \( P \) \((x,y)\), the slope of the rotating line, and product of the slope of \( \overline{AB} \) and the slope of the rotating line.
3. Drag A, B, and line k. The special relationship, \((x,y)\) and \((-y,x)\), between \( P \) and its rotated image, \( Q \), is relative to their intersection point. Hence it shows up clearly whenever the intersection point is at the origin.
4. At any time, press the “Reset” button to restore the screen to its initial state.

How to run this animation:
1. Press button #1. \( \overline{AB} \) is broken into its vertical and horizontal components and their intersection is graphed.
2. Press button #2. The vertical component’s midpoint is calculated using the average of the y-coordinates.
3. Press button #3. The horizontal component’s midpoint is calculated using the average of the x-coordinates.
4. Press button #4. These two midpoints merge to create the ordered pair of the midpoint of \( \overline{AB} \). The formula appears at the top of the screen.
5. One at a time, drag points A and B. The dynamic ordered pairs in the top right corner (along with all the other data) adjust for the movement.
6. If you want points A and B to have integer coordinates only, go to the top pull-down menu “Graph” and release on “Snap Points”. When dragged, they will jump to these spots.
7. At any time, press the “Reset” button to restore the screen to its initial state.

How to run this animation: Stress that \( \Delta x \) means “change in x” (coordinates), and \( \Delta y \) means “change in y”.
1. Press button #1. \( \overline{AB} \) is broken into its vertical and horizontal components and their intersection is graphed.
2. Press button #2. The vertical component’s length is calculated as the difference of the y-coordinates.
3. Press button #3. The horizontal component’s length is calculated as the difference of the x-coordinates.
4. Press button #4. The Pythagorean Theorem is applied to this right triangle. The square root of this equation yields the hypotenuse length, AB.
5. One at a time, drag points A and B. All of the dynamic coordinates & calculations adjust for the movement.
6. If you want points A and B to have integer coordinates only, go to the top pull-down menu “Graph” and release on “Snap Points”. When dragged, they will jump to these spots.
7. At any time, press the “Reset” button to restore the screen to its initial state.
"Make Open Box"

REMOVE SQUARE CORNERS FROM A RECTANGLE; FOLD; EXPLORE PAN'S VOLUME

Nearly every algebra class from pre-algebra through pre-calculus explores the volume of the open box (pan), each year revealing a new facet to this jewel of a problem. It is the quintessential “vertical teaming” concept. This animation adapts to each course and even addresses the calculus of the situation using derivatives and a tangent line. Choose the features (by pressing the appropriate buttons) suitable for your class.

Always press button #1 before any of the rest. Then choose any of buttons #2-5 as desired, skipping features that are beyond the scope of your particular class.

How to run this animation:
1. Press the “Show..” button. A text box appears explaining how to alter the length and width of the original rectangle and how to set a desired "x target" (x will be the length of the sides of the removed squares).
2. Press button #1. Repeat this button as often as necessary. Squares, with side x, are removed from the 4 corners of the rectangle. A copy of the altered rectangle drops flat and moves up to a free space on the screen. Its sides are folded up to form the box (pan) but a shadow of it remains underneath. A demonstration of how the original lengths “a” and “b” become a-2x and b-2x is shown. Finally the dynamic length, width, height, exterior surface area, and volume of the box appear on the screen.
3. To vary the size of the removed square, press button #2 OR Manually drag the red “Vary Sqs” OR set and move to a particular target value for x (the “Show..” button gives directions). In each case, the picture of the box and all of its data adjust accordingly.
4. Press button #3. Graphs of the box’s exterior surface area and volume, based on values of x, appear. If button #2 is not running, press it to see all the information dynamically move.
5. Press button #4 (calculus). A graph of the derivative of the volume with respect to changes in x is added.
6. Press button #5. Square side length x moves to the precise location for the maximum volume of the box.
7. At any time, press the “Reset” button to restore the screen to its initial state.

Suggestions:

For pre-algebra, begin with a square piece of material 8x8. Button #1 will begin with x = 2 units (always starts at ¼ of b) which is convenient for this level. Students also should experience making this pan out of paper.

For algebra 1, the first time through, follow the same procedure as described for pre-algebra above. However, then ask what would happen if the original material was rectangular. Conjecture. Then repeat the animation using the 8x6 original rectangle. Discuss domain restrictions on x. If there is time and the class is interested, you may want to proceed farther by asking for conjectures of where the maximum volume would occur and why. Have button #2 running, but scroll the window up so that the volume data at the bottom of the screen is obscured. Ask when to stop so that the box appears largest. Press button #5 to reveal the answer. Often the visual conjectures are not very accurate here – hence the need for an analytic approach, and a great discussion! (The problem seems to be that we tend to look for the box to become as cubical as possible. If the surface area were constant, that conjecture holds, but in this problem, the base dimensions are overly compromised for the sake of the height.) The graphs of the area and volume may also be of interest. The expression and graph for area could be explored once the students have studied quadratic functions. The expression for volume could be created, but its graph is left for a later class, as is button #4.

For algebra 2 and pre-calculus, begin with the initial 8x6 rectangle. Explore all buttons, except #4. When viewing the graph, discuss the cubic polynomial expression for volume and the local maximum of its graph. Discussions will be similar to those above, but of a greater depth for each successive course.

For calculus, explore all buttons in order. Of special interest is button #4. Note that the maximum volume occurs when the tangent is horizontal (slope 0). Of course, the graph of the derivative of the volume has a zero value here (crosses the x-axis) and it denotes a relative maximum volume since the derivative is positive prior to this point (above the x-axis) and negative afterwards (below the x-axis). But what about that minimum of the dV/dx graph? Sure it’s the fastest decrease in volume and the location of an inflection point for the volume graph, but is there another connection to where it occurs? YES! It’s easiest to discover it from a square original sheet and then generalize it to the rectangle. [Answer: x = (1/3)(average of a and b); when a=b, this yields a cube.]
“Multiply Polynomials”

GEOMETRIC MODELS FOR (a+b)c, (a+b)(c+d), (a+b)^2, (a+b)^3, (a+b+c)(d+e+f), a^2 – b^2

5 separate animations can be accessed from the page tabs (lower left of GSP screen):

- mult monomials & binomials
- expanding a square
- expanding a cube
- mult trinomials
- factoring a^2 - b^2

Dynamic area and volume of rectangles and cubes demonstrate the distributive property, F.O.I.L, and factoring.

How to run this animation:

Initially, the product of two monomials appears as the area of a rectangle. Drag the large red endpoints to alter. Each white diagonal demarcates the rectangular area associated with the measure written on it.

1. Press button #1. Side “a” is extended by “b” to form a side of length a+b. A large black frame appears around the product of (a+b)c which consists of two smaller rectangles, ac and bc. This demonstrates the distributive property: (a+b)c = ac + bc, written in the red box. Drag b’s red endpoint. When b’s red endpoint backs up (to the left) over the length of “a”, the area within the black frame shows that (a-b)c = ac - bc.
2. Press button #2. Similar to the results from button #1, it is shown that ±(a b)c ±(a b)(c d) ±(a ±b)2 ±(a ±b)(c ±d) ±(a ±b)b ±(a ±b)c ±(a ±b)2.
3. Press button #3. The product of two binomials is shown (F.O.I.L). Drag any large red points. The appropriate result appears in the red box. Avoid the situation (a-b)(c-d) if the double subtraction of area bd is beyond the scope of your particular class.
4. Press button #4. The vertical lengths adjust to match the horizontal lengths to see (a+b)^2 = a^2 + 2ab + b^2. Drag the large red points. Drag b to the left to subtract from “a” to show (a-b)^2 if appropriate for your class.
5. At any time, press the “Reset” button to restore the screen to its initial state.

“expanding a square”

Each side of a square begins at length x and expands by k units emphasizing that (x+k)^2 ≠ x^2 + k^2.

How to run this animation:

1. Press button #1. The turquoise x^2 area is constant. The pink (x+k)^2 area varies as k varies. Both of the red “drag” points may be moved manually instead.
2. Press button #2. A copy of the x^2 area appears inside the varying (x+k)^2 area for comparison.
3. Press button #3. Areas x^2 and k^2 appear inside the (x+k)^2 area to show that some area is still missing. The numerical areas also do not match. Stop and restart button #1 to compare these values.
4. Press button #4. Two xk areas fill the gaps so (x+k)^2 = x^2 + 2xk + k^2. Numerical areas also match.
5. Press either yellow button to see: side ratio = perimeter ratio; (side ratio)^2 = area ratio.
6. At any time, press the “Reset” button to restore the screen to its initial state.

“expanding a cube”

Each side of a cube begins at length x and expands by k units emphasizing that (x+k)^3 ≠ x^3 + k^3.

How to run this animation:

1. Press button #1. The turquoise x^3 volume is constant. The pink (x+k)^3 volume varies as k varies. Both of the red “drag” points may be moved manually instead.
2. Press button #2. A copy of the x^3 volume appears inside the varying (x+k)^3 volume for comparison.
3. Press button #3. Volumes x^3 and k^3 appear inside the (x+k)^3 volume to show that much volume is still missing. The numerical volumes also do not match. Stop and restart button #1 to compare these values.
4. Press button #4. One by one, the 3 x^2k volumes and the 3 xk^2 volumes fill the gaps so (x+k)^3 = x^3 + 3x^2k + 3xk^2 + k^3. Numerical areas also match.
5. Press either yellow button to see: (side ratio)^3 = volume ratio.
6. At any time, press the “Reset” button to restore the screen to its initial state.
**“mult trinomials”**

This animation geometrically addresses the products: \(a(d+e+f)\), \((a+b)(d+e+f)\), \((a+b+c)(d+e+f)\), and \((a+b+c)^2\).

**How to run this animation:**
Initially, a rectangle whose dimensions are \((a)\) and \((d+e+f)\) appears.
1. Press button #1. Two other buttons appear. Press the “step by step” button to see \(a(d+e+f)\), followed by \((a+b)(d+e+f)\), then \((a+b+c)(d+e+f)\). Press the “jump to end” button to see \((a+b+c)(d+e+f)\) as a final result only. To repeat, reset the screen to its initial state by re-pressing button #1 or pressing the “Reset” button.
2. Press button #2. \((a+b+c)^2\) appears with its 9 inner rectangles shaded but not labeled. Press the “Show answer” button to see the sum of the areas of these 9 inner rectangles.
3. Any red point labeled “drag..” can be dragged manually.
4. Press the “Reset” button to restore the screen to its initial state, *but be certain first to stop the “step by step” button, if it is running* (if it is running, it is darkened – repress any darkened button to stop it).

**“factoring a^2 – b^2”**

This animation provides a dynamic and visual geometric proof of why \(a^2 – b^2\) factors into \((a+b)(a-b)\).

**How to run this animation:**
Initially, a square with area \(a^2\) appears.
1. Press button #1. A growing square, with area \(b^2\), is removed from the upper left corner of \(a^2\).
2. Press button #2. A diagonal cuts the \(a^2 – b^2\) area into two congruent trapezoids (one is yellow, the other is gray). A duplicate of this situation also appears with all side lengths labeled. Make certain students understand the two black \((a-b)\) lengths.
3. Press button #3. In the duplicate, the gray trapezoid rotates \(90^\circ\) (it will slightly shrink, then stretch back as the rotation occurs – ignore this minor distortion), reflects about its median, and translates (slides) back to rejoin the yellow trapezoid. The two trapezoids rejoin to form a rectangle whose area is \((a+b)(a-b)\). Since both the left and right pictures contain the same two trapezoids, their areas are equal. Therefore, \(a^2-b^2 = (a+b)(a-b)\).
4. The red “drag a” and “drag b” points can be dragged manually.
5. At any time, press the “Reset” button to restore the screen to its initial state.
“Parabola with 3 Tangents”

ANY 3 TANGENTS TO ANY PARABOLA CUT EACH OTHER INTO PROPORTIONAL PARTS

3 separate animations can be accessed from the page tabs (lower left of GSP screen):

The animation explores and proves a very interesting relationship between any three tangents to any parabola.

How to run this animation:
Initially, a parabola is graphed whose equation is \( f(x) = 0.2(x-7)^2 + 5 \).
1. Press button #1. Three tangent lines appear at points A, B, and C. Conjectures are unlikely at this stage.
2. Press button #2. Two segments on each tangent line are highlighted. Ask for conjectures. If no changes were made, the larger segment on each tangent appears to be twice the smaller one. Drag A, B, C to see that the 2:1 ratio is not constant but ratios still appear to be equivalent to each other.
3. Press button #3. Relevant dynamic distances and ratios appear. The conclusion appears in a yellow text box along with an explanation of other ways to state this same result. Re-drag points A, B, C.
4. At any time, press the “Reset” button to hide all but the original data (values of r,h,k stay as they are).

Alterations to the parabola \( f(x) = r(x-h)^2 + k \):
1. To change r, double-click its value just below the blue function expression. A window opens. Enter desired value. Press “OK”.
2. To change vertex V(h,k), drag the large blue vertex point on the parabola. The dynamic coordinates for V adjust automatically.

“1st part of proof”
First, it is proved that \( \overline{RM_1} \parallel \overline{TM_2} \parallel \overline{SM_3} \) where \( M_1, M_2, M_3 \) are midpoints of the segments joining the points of tangency of the intersecting tangent lines.

How to run this animation:
1. Press button #1 to draw \( \triangle ABC \) and the midpoints of each of its sides.
2. Press button #2. A line is drawn from each midpoint to the green intersection point beneath it.
3. Press button #3. A body of text appears. It proves that the 3 lines drawn by button #2 are parallel:
   - Using point (a, f(a)) and slope \( f'(a) \), the equation of \( \overline{AR} \) is generated.
   - Using point (c, f(c)) and slope \( f'(c) \), the equation of \( \overline{RC} \) is generated.
   - By setting these two equations equal to each other and solving, the x-coordinate of point R (the intersection of the two lines) is found to be \( (a+c)/2 \). Since this is equivalent to the x-coordinate of the midpoint \( M_1 \), the line from \( M_1 \) to \( R \) is vertical.
   - Similar arguments (not shown) prove that the lines from \( M_2 \) to \( T \) and \( M_3 \) to \( S \) are vertical.
   - Since \( \overline{RM_1} \), \( \overline{TM_2} \), and \( \overline{SM_3} \) are each vertical, they are parallel.

“2nd part of proof”
Using various triangles and the 3 parallel lines proved in part 1, it is proved that
\[
\frac{AR}{RS} = \frac{ST}{TB} = \frac{RC}{CT}.
\]
The proof is detailed on the screen and color-coded for easier reading. Press the 2 buttons, when indicated, to highlight specific parts of the drawing. Press the “Reset” button to restore the screen to its initial state.
“Polynomial Surprises”

6 PROPERTIES OF POLYNOMIALS THAT MAY EVEN SURPRISE THE TEACHERS!

6 separate animations can be accessed from the page tabs (lower left of GSP screen):

Many of these properties will amaze and intrigue your students. A proof of each is included below.

1. In any parabola, the slope of any secant CD = slope of tangent to parabola at the midpoint of CD.

How to run this animation:
1. M is the midpoint of CD. Conjecture about CD and the parabola’s tangent at the same x-coordinate as M.
2. Drag red points V, C, D to test conjectures. (As V moves, its dynamic coordinates, h and k, adjust.)
3. Further test conjectures by changing r in the function: double-click the value of r in the upper right of the screen. A window opens. Type a new value. Press “OK”.
4. Press the “Show Conclusion” button (toggles into a “Hide” button).
5. For calculus classes, this is a special case of the Mean Value Theorem for Derivatives.

Proof:
Let f(x) = r(x-h)^2 + k Consider any points (a,f(a)) and (b,f(b))

slope of secant = f(b) - f(a) / b-a
= (rb^2 - 2rmb + rh^2 + k) - (ra^2 - 2rah + rh^2 + k) / b-a
= rb^2 - ra^2 - 2rh(b-a) / b-a
= r(b + a) - 2rh = rb + ra - 2rh

slope of f = f’(x) = 2r(x-h)
slope of f at secant’s mdpt = f’(a+b)/2 = 2r(a+b)/2 - h

= ra + rb - 2rh

same, Q.E.D.

2. In any cubic with a local minimum and local maximum, the midpoint of the segment joining them is the inflection pt.

How to run this animation:
1. (r,s) and (t,u) are local extrema. P is the midpoint of the segment joining them.
2. Conjecture about how midpoint P relates to the cubic.
3. Drag the red local minimum and local maximum points (they control the cubic) to test conjectures. As they move, the dynamic values of a,b,c,d,r,s,t,u adjust automatically.
4. Press the “Show Conclusion” button (toggles into a “Hide” button).

Proof:
Let (r,s) and (t,u) be local extrema of f(x) = ax^3 + bx^2 + cx + d

f’(x) = 3ax^2 + 2bx + c = 0 when x = -b + sqrt(b^2 - 4ac) / 2a
\therefore r = -b + sqrt(b^2 - 3ac) / 3a and t = -b + sqrt(b^2 - 3ac) / 3a

abscissa of mdpt = r + t / 2 = -b / 3a
For abscissa of inflection point,
f”(x) = 6ax + 2b = 0 when x = -b / 3a
\therefore abscissa of inflection point = abscissa of midpoint

Using TI89 and values for r & t at left,
ordinate of inflection pt. = f’’(b / 3a) = -bc / 3a + 2b^3 / 27a^2 + d
ordinate of mdpt = s + u / 2 = f(r) + f(t) / 2 = ar^3 + br^2 + cr + d + at^3 + bt^2 + ct + d

= ar^3 + tr^2 = b(t^2 + t^2) + c(r + t) / 2 + d

(TI89 with values of r & t at left) = -bc / 3a + 2b^3 / 27a^2 + d
\therefore ordinate of inflection point = ordinate of midpoint
Q.E.D.
"3"

In any cubic with 3 real roots, the tangent to the cubic at the average of any 2 roots will pass through the 3rd root.

How to run this animation:
1. (r,s) and (t,u) are local extrema. The point of tangency always has the same x-coordinate as point "mdpt".
2. Manually drag the red points R₁ and R₂ to any 2 roots (x-intercepts) of the graph of f(x).
3. Conjecture about where the tangent line intersects the graph of f(x).
4. Drag R₁ and R₂ to different roots of f(x) to test conjectures.
5. Further test conjectures by dragging the red local minimum and local maximum points (they control the cubic). As they move, the dynamic values of a,b,c,d,r,s,t,u adjust automatically. Loop back to step #2.
6. Press the "Show Conclusion" button (toggles into a "Hide" button).

Proof:
If f(x) has roots a,b,c then f(x) = (x-a)(x-b)(x-c) = x³ - (a+b+c)x² + (ab + ac + bc)x - abc
f'(x) = 3x² - 2(a+b+c)x + ab + ac + bc
Slope at midpoint M of roots a & b = f'(a + b) / 2 = -((a-b)² / 4);
Point M = [(a+b), -(a-b)²(a+b-2c)/8]
Equation of tangent line at M: y = -((a-b)²(a+b-2c)/8) / 4(x - a+b/2)
The tangent line’s x-intercept (set y = 0 and solve for x) is c. (c is third root) Q.E.D.

"4"

In any cubic with an inflection point, the ratio of the area bounded by any tangent to the cubic (intersecting the cubic again at W) to the area bounded by the line joining the inflection point to W is 27:16.

How to run this animation:
1. (r,s) & (t,u) are local extrema. The tangent line at P intersects f(x) again at W. Point I is the inflection point.
   The pink area is bounded by the tangent line and f(x). The green area is bounded by IW and f(x).
2. Conjecture about the ratio of the pink and green areas.
3. Manually drag the red point P to test conjectures.
4. Further test conjectures by dragging the red local minimum and local maximum points (they control the cubic). As they move, the dynamic values of a,b,c,d,r,s,t,u adjust automatically.
5. Press the "Show Conclusion" button (toggles into a "Hide" button). Again drag points P, (r,s), and (t,u).

Proof:
f(x) = ax³ + bx² + cx + d,  f'(x) = 3ax² + 2bx + c,  f''(x) = 6ax + 2b,  P at (p,f(p))
Let y₁ = eq. of tangent line at P → y₁ = f'(p)(x-p) + f(p) = (3ap² + 2bp + c)(x-p) + ap³ + bp² + cp + d
Set eq. of tangent line equal to f(x); solve, using TI89, to find intersection point W at (-2p - b/a, f(-2p - b/a))
f''(x) = 0 when x = -b/3a so inflection point I at (-b/3a, f(-b/3a))
Eq. of line IW → y₁ = f(-b/3a) + f(-b/3a)(x + b/3a)
Pink Area = ∫₀⁻²p-b/a (y₁ - f(x)) dx = -2p-b/a)[81a³p⁴ + 108a³b⁵ + 54a³b²p² + 12ab³p + b⁴]
Green Area = ∫₀⁻²p-b/a (y₁ - f(x)) dx = 4[81a³p⁴ + 108a³b⁵ + 54a³b²p² + 12ab³p + b⁴]
Pink Area  = 17  = 27
Green Area  = 4  81  = 16 Q.E.D.
In any cubic, consider a tangent line at any point P (intersects cubic again at W) and a tangent at W (intersects cubic again at Z). The ratio of the 2 areas bounded by the cubic and each of these lines is 16:1.

How to run this animation:
1. (r,s) & (t,u) are local extrema. The tangent line at P intersects f(x) again at W. The tangent line at W intersects f(x) again at Z. The pink and gray areas are bounded by one tangent line and f(x).
2. Conjecture about the ratio of the pink and gray areas. Manually drag the red point P to test conjectures.
3. Further test conjectures by dragging the red local minimum and local maximum points (they control the cubic). As they move, the **dynamic** values of a,b,c,d,r,s,t,u adjust automatically.
4. Press the “Show Conclusion” and “Show More Ratios” buttons (they toggle into a “Hide” buttons).

Proof:

\[ f(x) = ax^3 + bx^2 + cx + d, \quad f'(x) = 3ax^2 + 2bx + c, \quad f''(x) = 6ax + 2b + cp + d \]

Let y11 = eq. of tangent line at P \( y_{11} = f'(p)(x - p) + f(p) \)

\[ y_{11} = 3a(p)^2 - 2bp + c \]

Set eq. of tangent line \( y_{11} \) equal to f(x); solve, using TI89, to find intersection point W at \( \left(-2p + \frac{b}{a}, -2p + \frac{b}{a}\right) \)

Let pink area be “above” f and Z “left” of P “left” of W.

\[ \text{Pink Area} = \int_{-2p - \frac{b}{a}}^{2p + \frac{b}{a}} (y_{11} - f(x)) \, dx = \frac{4(81a^4p^4 + 108a^2bp^3 + 54a^2b^2p^2 + 12ab^3p + b^4)}{12a^3} \] (T89)

\[ \text{Gray Area} = \int_{-2p - \frac{b}{a}}^{2p + \frac{b}{a}} (f(x) - y_{11}) \, dx = \frac{4(81a^4p^4 + 108a^2bp^3 + 54a^2b^2p^2 + 12ab^3p + b^4)}{12a^3} \] (T89)

\[ \text{Pink Area} : \text{Gray Area} = \frac{4}{3} : 1 = 16 \, \text{Q.E.D.} \]

Simple alterations to the limits of integration prove the 3 additional ratios shown in the animation.

For any quartic with 2 inflection pts \( i_1 \) & \( i_2 \), the ratios of the 3 areas bounded by the quartic and \( \frac{i_1}{i_2} \) is 1:2:1.

How to run this animation:
1. (r,s) & (t,u) are local extrema. The line joining inflection points \( i_1 \) and \( i_2 \) intersects f(x) again at K and M.
2. Conjecture about the 3 areas bounded by \( \frac{i_1}{i_2} \) and f(x). (Note that the 2 blue regions are NOT congruent.)
3. Manually drag the red points (r,s), (t,u) and (v,?) to test conjectures. Dynamic values adjust automatically.
4. Press the “Show Conclusion” button (toggles into a “Hide” button). Again drag points (r,s), (t,u), (v,?).

Proof:

\[ f(x) = ax^4 + bx^3 + cx^2 + dx + e, \quad f'(x) = 4ax^3 + 3bx^2 + 2cx + d, \quad f''(x) = 12ax^2 + 6bx + 2c \]

\[ f'''(x) = 0 \text{ when } x = \frac{-3b + \sqrt{9b^2 - 24ac}}{12a} \] so inflection points at \( i_1 = \frac{-3b - \sqrt{9b^2 - 24ac}}{12a}, \quad i_2 = \frac{-3b + \sqrt{9b^2 - 24ac}}{12a} \)

Let \( y_i \) = eq. of line joining \( i_1 \) and \( i_2 \) \( y_i = \frac{f(i_1) - f(i_2)}{i_1 - i_2} (x - i_1) + f(i_1) \)

Set eq. of \( y_i \) equal to f(x); solve, using TI89, to find intersection points: K at \( \frac{-3b - \sqrt{45b^2 - 120ac}}{12a} \), M at \( \frac{-3b + \sqrt{45b^2 - 120ac}}{12a} \)

\[ \text{Left Area} = \int_{i_1}^{K} (y_i - f(x)) \, dx = \frac{64a^2c^2 - 48abc^2 + 9b^4}{8640a^4} \sqrt{9b^2 - 24ac} \] (T89)

\[ \text{Middle Area} = \int_{i_1}^{M} (f(x) - y_i) \, dx = \frac{64a^2c^2 - 48abc^2 + 9b^4}{4320a^4} \sqrt{9b^2 - 24ac} \] (T89)

\[ \text{Right Area} = \int_{i_2}^{M} (y_i - f(x)) \, dx = \frac{64a^2c^2 - 48abc^2 + 9b^4}{8640a^4} \sqrt{9b^2 - 24ac} \] (T89)

\[ \text{Left Area} : \text{Middle Area} : \text{Right Area} = 1:2:1 \, \text{Q.E.D.} \]
“Pythagorean Theorem”

EXPLORATION OF THIS “GREAT THEOREM” AND HISTORICAL NOTES

2 separate animations and a history page can be accessed from the page tabs (lower left of GSP screen):

$squares \text{ on sides}$ $\quad other \text{ areas on sides} \quad a \text{ bit of history}$

$squares \text{ on sides}$

This first animation explores the areas of squares on the sides of a right triangle and presents Pythagorean triples.

How to run this animation:
1. Press the “Step-By-Step” button. The initial triangle is a 3-4-5. Pausing between steps, squares are built on the sides of the right triangle and their dynamic areas appear. The grid behind the coordinate plane allows confirmation of the lengths of the legs and the areas of their squares. The square on the hypotenuse is rotated so its side length and area also can be confirmed on the grid. Ask students to conjecture the relationship of the areas if they are new to this remarkable theorem.
2. Press the “Show Theorem” button. All data appears at once, a statement of the theorem appears using words, not merely letters, and the sum of two areas is shown to equal the third.
3. Press the “Animate A,B” button. Points A & B slide along the axes. All dynamic data adjust automatically. Points A and B also can be dragged manually.
4. For three of the classic Pythagorean triples (integral right triangle lengths), press any of the 3 brown buttons.
5. At any time, press the “Reset” button to restore the screen to its initial state.

Rounded decimals can be misleading: When measuring, GSP uses a large number of digits but rounds off the displayed values. This may at times cause a calculation to appear to be off by one unit in the last digit. For example, if $2.3 + 1.4 = 3.6$ appears, the numbers may have been $2.26\ldots + 1.37\ldots$ which is $3.63\ldots$. 

$other \text{ areas on sides}$

This animation demonstrates that the special summed area relationship of the Pythagorean Theorem holds for any similar shapes built on the sides of a right triangle, not only squares.

How to run this animation:
1. Discuss the 4 right triangles that are shown. Each has geometrically similar shapes built on its sides: one has equilateral triangles, one has regular (equilateral and equiangular) pentagons, one has semicircles, and the classic one has squares.
2. Press the “Animate Points” button. The top vertex of each right triangle moves up and down, constantly changing the shape of its right triangle. The dynamic data shows the sum of all 4 blue areas, the sum of all 4 green areas, the sum of all blue and green areas combined, and the sum of all 4 red areas. The theorem still holds. To stop the movement, re-press the darkened “Animate Points” button.

“ladder”

This animation simulates a ladder leaning on the wall, slides it, and offers a few new “slants” to this classic favorite.

How to run this animation: Be sure students notice the changing speed of the ladder's height as it slides.
1. Follow steps #1-2 on the screen to set the ladder length and dist. from the wall. The wall height is 20 units.
2. Press the “Slide Ladder” button at any time. Re-press to stop. Drag base of ladder manually from red point.
3. With desired base dist. from wall, find the ladder’s height. Confirm by pressing the “Show Height” button.
4. Discuss the question in step #4 on the screen. Conjecture. Press “Answer”. A rag is revealed at the midpoint of the ladder and the ladder slides, leaving a fading trace of the rag’s path. Surprise! Stop the motion (re-press the button). Move the rag off the ladder’s midpoint. Conjecture. Press “Answer” again.
5. Discuss the question in step #5 on the screen. Conjecture. Press “Answer”. Graph appears; ladder slides.
6. At any time, press the “Reset” button to restart, however the ladder length & base dist. will stay the same.

For a tiny taste of the fascinating history of Pythagoras and his followers, press the page tab on the bottom of the GSP screen labeled “a bit of history”.

Use of the information on this page is restricted by the disclosure statement on the cover of this packet. amweeks@aol.com www.calculusinmotion.com
“Trigonometry”

FULLY EXPLORE THE UNIT CIRCLE, BASIC DEFINITIONS, AND THEOREMS

7 separate animations can be accessed from the page tabs (lower left of GSP screen):

- unit circle angles
- defs sin cos tan
- coords of unit circle
- unwrapping unit circle
- Pythagorean identities
- law of sines
- law of cosines

"unit circle angles"

This animation dynamically measures angles (degrees and radians) around the unit circle for up to 3 full revolutions in either a positive or negative direction. Basic vocabulary is shown and radian measure is defined.

How to run this animation:
1. Either drag theta manually on its number line, or press the “Animate theta slowly from 0” button. Note that when theta < 0 or theta > 360°, that value is displayed at the left, but the displayed angle’s measure will be its positive counterpart between 0° and 360°. Press the “Move theta to 0” button to put theta at 0.
2. Alternatively, set a target value for theta (from -1080° to 1080°) by double-clicking the red “Target Degrees” value. A window opens. Enter desired value. Press “OK”. Then press the red “Rotate to Target” button.

Note: An equivalent definition of 1 radian is the measure of the angle between two radii of a circle that cut off an arc of the circle equal in length to the radius.

"defs sin cos tan"

This animation defines the sine, cosine, and tangent ratios and shows why they are constant for any angle.

How to run this animation:
1. Press any of the 3 “Hide..” buttons to mix and match shown ratios (they toggle into “Show..” buttons).
2. Drag point A. The triangle changes size, not shape (remains similar) so ratios of sides are unchanged.
3. Drag point B. The triangle changes shape so ratios of sides change.
4. To set a specific length for AC and BC, double-click either measure in the upper right of the screen. A window opens. Enter desired value. Press “OK”. Finally, press the “Apply” button to apply these lengths.

"coords of unit circle"

Sine and cosine values, in radical form, for special angles \(\left\{\text{multiples of } \frac{\pi}{2}, \frac{3\pi}{4}, \frac{\pi}{6}\right\}\) are explained and rehearsed.

How to run this animation:
1. Discuss the gray text under the yellow title box - each point of the unit circle has coordinates \((\cos \theta, \sin \theta)\).
2. Press any of the 12 blue buttons (4 for each quadrant) labeled with angle measures. \(\angle\theta\)’s terminal ray rotates into that position. Dynamic measures are shown for the angle, its ordered pair intersection on the unit circle, and its cosine and sine (both in exact and rounded forms). P can also be dragged manually.
3. For a continuous sweep of the unit circle, press the “Rotate P (nonstop) button.
4. For one sweep of the unit circle, pausing at each special angle, press the “Once Around” button.

This animation provides an excellent opportunity to practice stating the trigonometric ratios for special angles.
"unwrapping unit circle"

A point moves around the unit circle. Simultaneously, its distance traveled vs. its x- or y-coordinate is graphed.

How to run this animation:
1. First explore the “Ferris Wheel” animation in the “Cycloids, Sq Wheels, Ferris” file, if desired.
2. For this animation, review what is written in the gray box. Discuss the questions posed in the yellow box.
3. Press the "Animate P" button. Conjecture about the graphs. (Have students attempt to draw them?)
4. Press the “Reset” button. Press the “Show Sine” or “Show Cosine” button (they toggle into “Hide” buttons).
5. Re-press the “Animate P” button. The selected graph (or both) is dynamically plotted. Stop and restart the motion by re-pressing this button. If desired, use the Motion Controller (from pull-down “Display” menu at top of screen) to change speed. P also can be dragged manually.

"Pythagorean identities"

This animation dynamically explores and proves the three Pythagorean Identities.

How to run this animation:
1. Review the text and images on the initial screen.
2. Press the “Show Identity #1” button to see $(\sin \theta)^2 + (\cos \theta)^2 = 1$ plus a proof for all circles (not just unit).
3. Press the “Show Identity #2” button to see $(\tan \theta)^2 + 1 = (\sec \theta)^2$ plus a proof for all circles (not just unit).
4. Press the “Show Identity #3” button to see $(\cot \theta)^2 + 1 = (\csc \theta)^2$ plus a proof for all circles (not just unit).
5. To move P: drag it manually, press the “Rotate P (nonstop)” button, or press the “Move P ->0” button.

"law of sines"

This animation dynamically explores and proves the Law of Sines.

How to run this animation:
1. Stress that this no longer need be a right triangle. From the data on the initial screen, ask for conjectures. (Each ratio can be estimated at just less than 1/11.)
2. Press the red “Show Result” button to test conjectures. The theorem and dynamic ratio values appear.
3. Drag points A, B, and C to further test this property. All dynamic data adjusts automatically.
4. Press either “Prove..” button to see a proof. Drag point B to further confirm. For the proof, if ΔABC becomes obtuse, be careful to continue to use RIGHT triangles when generating the sine ratios. Remember, sin x° = sin (180-x)°.
5. At any time, press the “Reset” button to restore the screen to its initial state.

"law of cosines"

This animation dynamically explores and proves the Law of Cosines.

How to run this animation:
1. Stress that this no longer need be a right triangle. From the data on the initial screen, it is hard to make a conjecture other than that the Pythagorean Theorem doesn’t apply.
2. Press button #1. An altitude appears creating 2 right triangles. What might be applied?
3. Press button #2. The Pythagorean Theorem is applied to each right triangle. The equations are merged.
4. Conjecture about answers to the green question on screen (need only original sides or angles in formula).
5. Press button #3. The substitution is made and the theorem appears along with the rest of the data.
6. Drag points A, B, and C to further test this property. All dynamic data adjusts automatically. If the altitude is exterior to the triangle, carefully watch the changes in lengths and be careful to continue to use RIGHT triangles to generate the cosine ratio. Remember, $\cos x^\circ = - \cos (180-x)^\circ$.
7. At any time, press the “Reset” button to restore the screen to its initial state.